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Statistical features of the evolution of two-dimensional turbulence

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Statistical fluid dynamics identifies a functional of the fluid energy spectrum that plays the role of Boltzmann's entropy for fluids. Through a series of two-dimensional flow simulations we confirm the theoretical predictions for the behaviour of this entropy functional. This includes a demonstration of Loschmidt's paradox and an examination of the effects of Rossby waves and viscosity on the behaviour of the entropy.

1. Introduction

Inviscid spectrally truncated flow represents a conservative system for which the methods of canonical statistical mechanics apply. In particular, a canonical equilibrium spectrum determined solely by the dynamical invariants of the system can be defined (Kraichnan 1967; Basdevant & Sadourny 1975; Salmon, Holloway & Hendershott 1976). It was first pointed out by Cook (1974) that for two-dimensional, incompressible, homogeneous flow the Markovian equations for the statistical evolution of the energy spectrum imply monotonic relaxation to canonical equilibrium (i.e. an H-theorem). Subsequently it has been shown that this result generalizes to all macroscopic fluid systems (Montgomery 1976; Carnevale 1979; Carnevale, Frisch & Salmon 1981). The theoretical importance of this work is that it applies to all Liouvillian systems independent of a Hamiltonian formulation, thus extending the use of H-theorems (Carnevale *et al.* 1981). Here we investigate how these results are manifested in simulations of two-dimensional flow.

The quantity that plays the central role in these studies is the functional

$$S \equiv \tfrac{1}{2}\sum_{\mathbf{k}} \ln U_{\mathbf{k}}, \tag{1.1}$$

where $U_{\mathbf{k}}$ is the ensemble average modal energy in wavevector mode $\mathbf{k}$. Specifically, $U_{\mathbf{k}}$ is given by

$$U_{\mathbf{k}} \equiv \langle \mathbf{v}_{\mathbf{k}} . \mathbf{v}_{\mathbf{k}}^{*} \rangle = k^2 \langle |\psi_{\mathbf{k}}|^2 \rangle, \tag{1.2}$$

where $\psi_{\mathbf{k}}$ is the Fourier-transformed stream function, and the angle brackets represent an average over an ensemble of initial states.

That a functional of the form S can be considered the entropy of macroscopic fluid motions was first suggested by Betchov (1964). He argues that this S is a measure of the information necessary to specify the turbulent motion of a fluid with spectrum $U_{\mathbf{k}}$. In a more general formulation Carnevale *et al.* (1981) have demonstrated a connection between the usual Gibbs prescription for information and the functional S. In the present context the connection can be briefly described as follows. Consider an ensemble of incompressible, two-dimensional flows with probability density

$P(\{\psi_{\mathbf{k}}\})$, where $\{\psi_{\mathbf{k}}\}$ includes all independent wavevector amplitudes. The Gibbs prescription for the amount of information contained in the distribution $P(\{\psi_{\mathbf{k}}\})$ is

$$H_G(P) = \int P \ln P,$$

with the integral taken over all elements of $\{\psi_{\mathbf{k}}\}$ (cf. Tolman 1938). If P is varied over all possible distributions with the same spectrum $U_{\mathbf{k}}$, then the minimum value that $H_G(P)$ assumes is, up to additive constants, just

$$\min H_G(P) = -\tfrac{1}{2}\sum_{\mathbf{k}} \ln U_{\mathbf{k}}.$$

In this sense then this lower bound on H_G is interpreted as the amount of information contained in a knowledge or specification of the spectrum $U_{\mathbf{k}}$ alone. Thinking of entropy as a lack of information leads to the prescription (1.1). For details and further discussion see the above references.

For inviscid two-dimensional flow there are two quadratic dynamical invariants: the total energy

$$E_T = \tfrac{1}{2}\sum_{\mathbf{k}} k^2|\psi_{\mathbf{k}}|^2, \tag{1.3}$$

and the total enstrophy

$$Z_T = \tfrac{1}{2}\sum_{\mathbf{k}} k^4|\psi_{\mathbf{k}}|^2. \tag{1.4}$$

These constraints determine the form of the canonical equilibrium spectrum (Kraichnan 1967)

$$U_{\mathbf{k}}^{\mathrm{eq}} = \frac{1}{a+bk^2}, \tag{1.5}$$

where a and b are constants defined by the values of E_T and Z_T. It can be shown that the maximum value that S can have for given E_T and Z_T is just that value computed with the equilibrium spectrum (1.5) (Carnevale *et al.* 1981).

It has been demonstrated previously (Cook 1974; Carnevale *et al.* 1981) that second-order Markovian closure theory implies that S evolves according to

$$\frac{dS}{dt} = \frac{1}{3}\sum_{k+p+q=0} U_{\mathbf{k}} U_{\mathbf{p}} U_{\mathbf{q}} \theta_{\mathbf{kpq}} \left(\frac{\sin(\mathbf{p},\mathbf{q})}{k}\right)^2 \left[\frac{p^2-q^2}{U_{\mathbf{k}}} + \frac{q^2-k^2}{U_{\mathbf{p}}} + \frac{k^2-p^2}{U_{\mathbf{q}}}\right]^2 - \nu\Sigma k^2. \tag{1.6}$$

All summations include only wavevectors of magnitude less than or equal to finite $k_{\max}$. $\theta_{\mathbf{kpq}}$ is a positive function called the triad relaxation time (Rose & Sulem 1978) which can be modelled in several ways. For inviscid spectrally truncated flow we have $dS/dt \geqslant 0$, and $dS/dt = 0$ if and only if $U_{\mathbf{k}}$ is given by (1.5). That is, for inviscid truncated flow the entropy is stationary only in the state of canonical equilibrium and otherwise increases monotonically.

Numerical simulation of inviscid two-dimensional flow exhibits relaxation toward the canonical spectrum (1.5) (Fox & Orszag 1973; Seyler *et al.* 1975). During such a simulation we follow the evolution of a single realization entropy defined by

$$S_e = \tfrac{1}{2}\sum_{\mathbf{k}} \ln(k^2|\psi_{\mathbf{k}}|^2), \tag{1.7}$$

and we compare its behaviour with the theoretical prediction for S_e based on an ensemble. The results on inviscid equilibrium in §2 emphasize the close correspondence between canonical statistical prediction and inviscid simulation. In §3 the statistical nature of the theory is further emphasized by a demonstration of Loschmidt's paradox, which strikingly displays the invariance of inviscid dynamics under time reversal. All this is in the same spirit as simulations of particle dynamics (cf. Orban & Bellmans 1967).

In §§4 and 5 the effects of Rossby-wave propagation and viscosity on the evolution of S_e are compared qualitatively with the predictions of closure theory, and quantitatively with canonical statistics.

2. Inviscid two-dimensional flow

Our simulations are produced with a de-aliased spectral code that conserves energy and enstrophy very accurately. Wavevectors have integral components, and we apply a circular truncation at wavenumber $k_{max} = 120$. The total number of modes simulated is $f = 45224$.

The initial state for these simulations is randomly generated, with approximately equal energy in each unit wavenumber band for $69 \leqslant k \leqslant 100$; that is

$$E(k) \equiv \tfrac{1}{2} \sum_{k \leqslant p < k+1} p^2 |\psi_p|^2 = \text{const} \quad (69 \leqslant k \leqslant 100).$$

The total energy is $E_T \approx 0{\cdot}3434$ and the total enstrophy is $Z_T \approx 2449$ in the arbitrary units of the simulation.

For these values of energy and enstrophy the corresponding canonical equilibrium spectrum is approximate energy equipartition (cf. Fox & Orszag 1973):

$$U_k^{eq} \approx \frac{1}{a} \quad \text{or} \quad E(k) \approx \frac{\pi k}{a}, \tag{2.1}$$

where $1/a = 2E_T/f$. Thus the equilibrium entropy is

$$S^{eq} \approx -\tfrac{1}{2} f \ln a \approx -250882. \tag{2.2}$$

It is useful to introduce the 'local' eddy turnover time

$$\tau_e(k) = (k^3 E(k))^{-\frac{1}{2}}. \tag{2.3}$$

The smallest τ_e in these simulations is found at high k and is roughly $0{\cdot}01$ (again in these arbitrary units).

Figures 1–4 show various stages in the evolution of the energy spectrum. Notice in figure 3 at time $t = 0{\cdot}2$ (or after roughly 20 of the fast eddy turnover times) that most of the high-wavenumber spectrum is near its equilibrium form, and this actually accounts for the bulk of the simulated modes. In figure 4 at time $t = 1{\cdot}5$ many more of the wavenumber bands have equilibrated. The few modes still far from equilibrium ($k \lesssim 5$) at $t = 1{\cdot}5$ will take considerably longer to relax (cf. Fox & Orszag 1973).

Figure 5 shows the evolution of S_e in this simulation. While the system is far from equilibrium S_e increases rapidly and monotonically, with significant change occurring in periods of the order of the fast eddy turnover timescale. As the higher-wavenumber modes begin to equilibrate, this increase slows rapidly. S_e continues to increase monotonically until around $t = 0{\cdot}14$, after which time minor fluctuations, which are imperceptible on the scale of figure 5, appear. On the enlarged scale of figure 6 we display these fluctuations for $0{\cdot}2 \leqslant t \leqslant 1{\cdot}5$. The time average of S_e over the timespan of figure 6 is

$$\bar{S}_e = -263985, \tag{2.4}$$

and the fluctuations are less than $0{\cdot}08\,\%$ of this value.

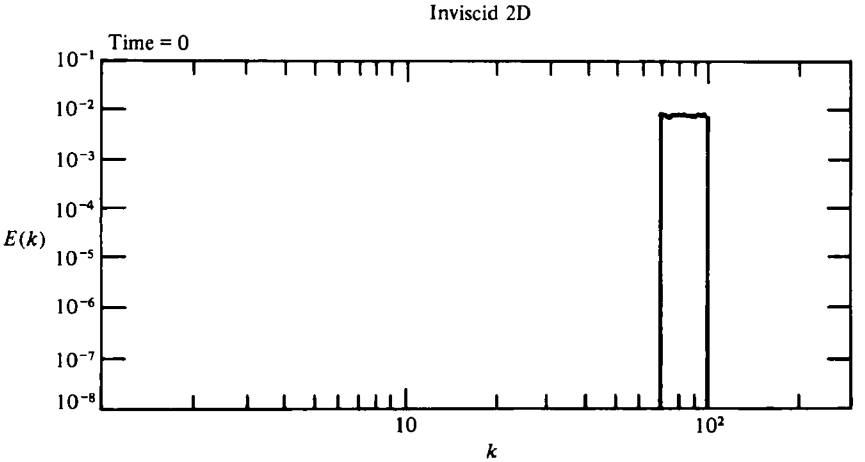

FIGURE 1. Randomly generated initial energy spectrum.

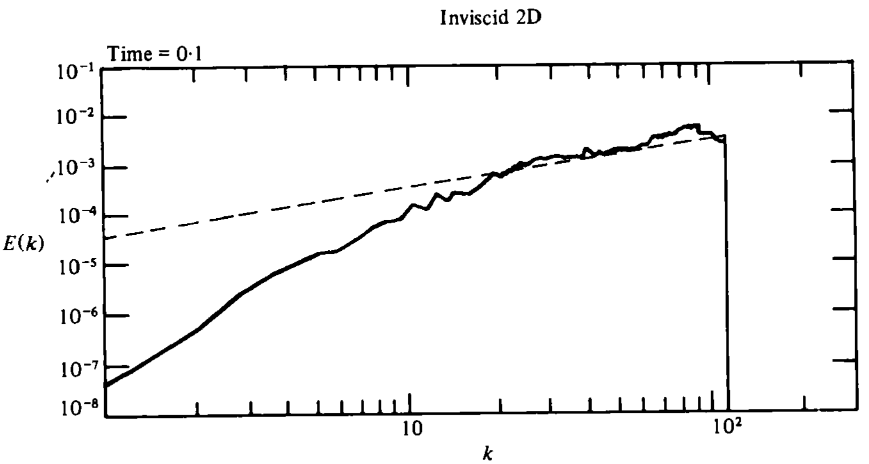

FIGURE 2. Energy spectrum in inviscid simulation at time $t = 0{\cdot}1$. Dashed line represents inviscid equilibrium.

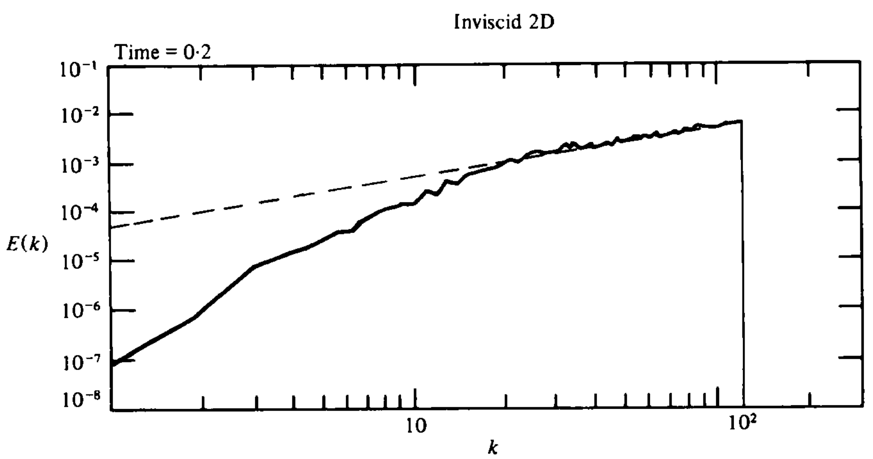

FIGURE 3. Energy spectrum in inviscid simulation at time $t = 0{\cdot}2$.

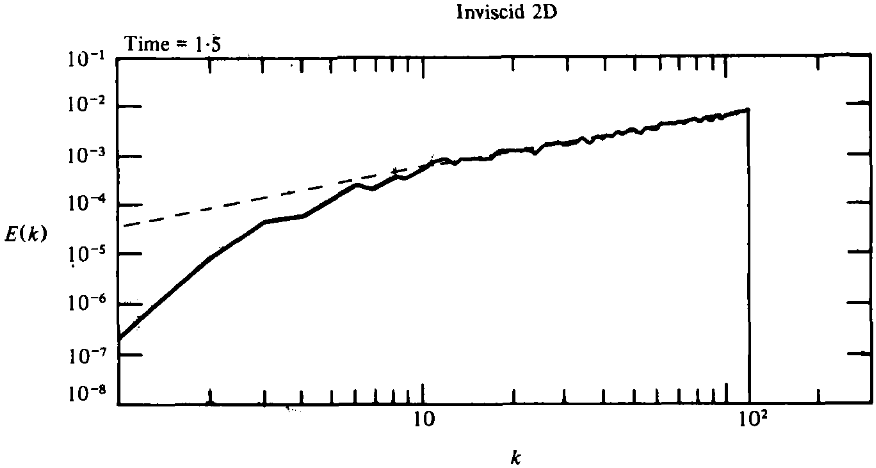

FIGURE 4. Energy spectrum in inviscid simulation at time $t = 1{\cdot}5$.

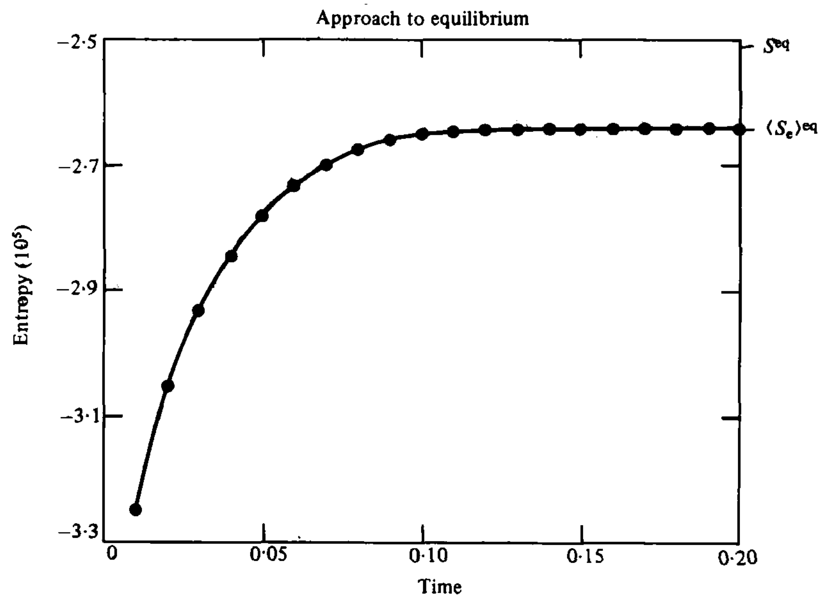

FIGURE 5. Inviscid evolution of entropy

$S_e \equiv \frac{1}{2}\sum_{\mathbf{k}} \ln k^2|\psi_{\mathbf{k}}|^2$. $S^{eq} \equiv \frac{1}{2}\sum_{\mathbf{k}} \ln k^2\langle|\psi_{\mathbf{k}}|^2\rangle^{eq} = -250882$. $\langle S_e\rangle^{eq} = -263934$.

To compare statistical theory with experiment we first remark that S^{eq} given by (2.2) is based on an ensemble-average energy spectrum $U_{\mathbf{k}}^{eq} = \langle k^2|\psi|^2\rangle^{eq}$, and we cannot then expect

$$S_e = \tfrac{1}{2}\sum_{\mathbf{k}} \ln k^2|\psi_{\mathbf{k}}|^2,$$

which is based on a single realization of $\psi_{\mathbf{k}}$, to achieve the value S^{eq}, no matter how long the simulation is run. The value of S_e when near equilibrium should properly be compared with the expectation value $\langle S_e\rangle^{eq}$. The equilibrium average is taken with the canonical distribution

$$P(\psi_{\mathbf{k}}) \propto \exp\{-\Sigma k^2|\psi_{\mathbf{k}}|^2 a\}, \tag{2.5}$$

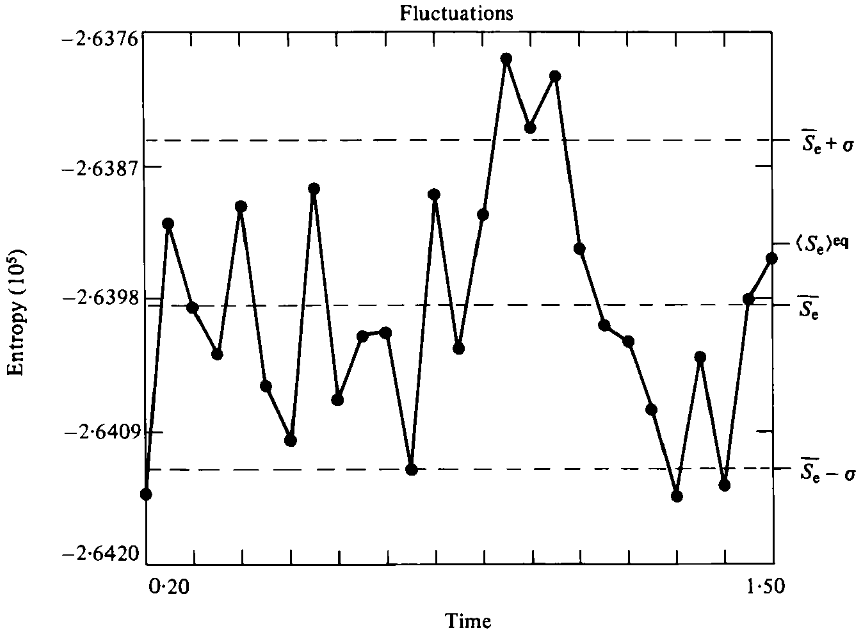

FIGURE 6. Fluctuations in entropy in near-equilibrium state. The time average $\bar{S}_e = -263985$; the predicted standard deviation $\sigma = (\langle (S_e - \langle S_e \rangle)^2 \rangle)^{\frac{1}{2}} = 136$.

where again $1/a = 2E_T/f$. The result is

$$\langle S_e \rangle^{eq} = S^{eq} + \tfrac{1}{2} f\gamma$$
$$= -263934, \tag{2.6}$$

(γ is Euler's constant). Note that $\langle S_e \rangle^{eq}$ compares favourably with the time average $\bar{S}_e$ computed from the simulation. That $\bar{S}_e$ is only 0·02 % less than $\langle S_e \rangle^{eq}$ can easily be because of the fact that even at $t = 1{\cdot}5$ the energy in the lower wavenumbers still has not equilibrated. If it were feasible to run the simulation to much longer times, then assuming ergodic behaviour (cf. Basdevant & Sadourny 1975) we would expect a later time average of S_e to come even closer to $\langle S_e \rangle^{eq}$. Furthermore, canonical statistics yield an estimate of the size of the fluctuations in S_e. Computing the standard deviation based on the distribution (2.5) we obtain

$$\sigma = (\langle (S_e - \langle S_e \rangle)^2 \rangle)^{\frac{1}{2}} \approx 136. \tag{2.7}$$

In figure 6 we see that $\bar{S}_e \pm \sigma$ does indeed give a good fit to the fluctuation range. For clarity and emphasis we repeat these results in table 1.

3. Loschmidt's paradox

The accuracy of the statistical predictions is remarkable when we emphasize that we are comparing them to a single realization, not an ensemble. We could certainly imagine initial conditions for which the entropy would behave contrary to the statistical predictions by displaying so-called antikinetic behaviour. For example, at any point in the evolution we could create an antikinetic state by reversing the velocity field. Since the inviscid dynamics are time-reversible the evolution of this new state would proceed not toward equilibrium but 'backwards' towards the initial

Quantity	Definition	Value
S^{eq}	$\frac{1}{2}\sum_{\mathbf{k}} \ln\langle k^2\lvert\psi_{\mathbf{k}}\rvert^2\rangle^{\mathrm{eq}}$	−250882
$\langle S_e\rangle^{\mathrm{eq}}$	$\langle\frac{1}{2}\sum_{\mathbf{k}} \ln k^2\lvert\psi_{\mathbf{k}}\rvert^2\rangle^{\mathrm{eq}}$	−263934
$\bar{S}_e$	$\frac{1}{27}\sum_{n=0}^{26} S_e(t = 0{\cdot}2+0{\cdot}05n)$	−263985
σ^2	$\langle(S_e-\langle S_e\rangle)^2\rangle$	136^2

TABLE 1.

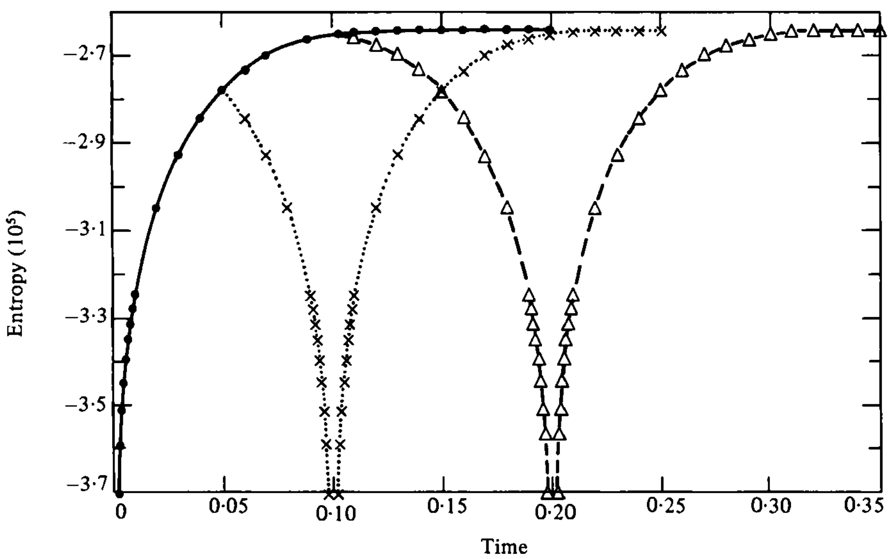

FIGURE 7. Demonstration of Loschmidt's paradox. The Loschmidt demon reverses the velocity field. In simulation represented by short (long) dashed curve the demon acts at $t = 0{\cdot}05$ ($t = 0{\cdot}10$).

state with reversed velocities. Thus the entropy must decrease monotonically until it achieves its initial value; subsequent evolution should then again show evolution of S_e towards equilibrium. The apparent disparity between statistical theory and this antikinetic behaviour is called Loschmidt's paradox (Cercignani 1975). It is resolved by realizing that the H-theorem is a statistical statement and antikinetic states form a set of measure zero when compared with all possible states with the same energy spectrum.

In figure 7 we provide a demonstration of Loschmidt's paradox. At times $t = 0{\cdot}05$ and again at $t = 0{\cdot}10$ in the simulation of §2, we conjure up a demon that reverses the velocity field, and then we proceed simulating forward in time. The result is just as time-reversibility would suggest. In the first stage after the demon intervenes the history of S_e is reproduced with six-significant-figure accuracy. This is a very encouraging indication of the precision of these simulations. Subsequent evolution shows S_e increasing towards its expectation value, again with fluctuations in the predicted noise range.

4. Inviscid homogeneous flow on the β-plane

Flow on a β-plane is a flat-geometry model for flow on a rotating sphere (Rhines 1975). The presence of a variable Coriolis force permits the propagation of Rossby waves with frequency $\omega_{\mathbf{k}} = -\beta k_x/k^2$ (where β is the gradient of the Coriolis force). Some simple theoretical arguments suggest that the presence of these wave modes will retard the transfer of energy from wavenumbers greater than the Rhines wavenumber $k_\beta = (\beta/2V)^{\frac{1}{2}}$, where V is an r.m.s. velocity, to lower wavenumbers (Rhines 1975). Furthermore a tendency toward anisotropic (zonal) flow is expected while the system is far from equilibrium (Rhines 1975; Holloway & Hendershott 1977), although the ultimate equilibrium spectrum must still be the isotropic spectrum (1.5), because even on the β-plane the quadratic invariants are still E_{T} and Z_{T} (Carnevale *et al.* 1981).

In the Markovian closure equations for homogeneous flow the effect of wave propagation enters only implicitly through the values of the triad relaxation time $\theta_{\mathbf{kpq}}$ (Holloway & Hendershott 1977; Legras 1980; Carnevale & Martin 1982). For example, a convenient expression for $\theta_{\mathbf{kpq}}$ in pure two-dimensional inviscid flow is the eddy-damped quasi-normal model

$$\theta_{\mathbf{kpq}} = \frac{1}{\mu_{\mathbf{k}} + \mu_{\mathbf{p}} + \mu_{\mathbf{q}}}, \tag{4.1}$$

where μ_k is the eddy turnover rate. For β-plane flow this must be modified to

$$\theta_{\mathbf{kpq}} = \frac{\mu_{\mathbf{k}} + \mu_{\mathbf{p}} + \mu_{\mathbf{q}}}{(\omega_{\mathbf{k}} + \omega_{\mathbf{p}} + \omega_{\mathbf{q}})^2 + (\mu_{\mathbf{k}} + \mu_{\mathbf{p}} + \mu_{\mathbf{k}})^2}, \tag{4.2}$$

(Holloway & Hendershott 1977). This suggests that for a given energy spectrum the energy transfer rate is smaller and more anisotropic the larger the value of β. Equation (1.6) implies that $dS/dt \geqslant 0$ also on the inviscid β-plane, and that we must expect a slower rate of equilibration the larger the value of β.

We have performed inviscid simulations with values of $\beta = 500$ and 1000 using the same initial conditions as in §2. These values of β correspond to Rhines wavenumbers of $k_\beta = 17$ and 25 respectively. Figure 8 demonstrates that the behaviour of S_{e} is qualitatively as anticipated. The strong values of β slow down the approach to equilibrium. Figure 9, when compared with figure 3, clearly shows the retarded transfer of energy through the soft barrier at k_β. A measure of the anisotropic tendencies is $v_x^2 - v_y^2 = \Sigma_{\mathbf{k}}(k_y^2 - k_x^2)|\psi_k|^2$. For the $\beta = 1000$ simulation ($t = 0$ to $t = 0{\cdot}2$) this measure, initially small and negative, goes positive and increases in size by an order of magnitude, clearly showing the far-from-equilibrium tendency towards zonal motion.

5. Viscous decay

According to (1.6) there are two competing effects in the evolution of S for viscous flow. Viscous decay contributes the negative constant $-\nu\Sigma k^2$, which in our simulations is $-\nu(325\,519\,928)$. From a statistical–information standpoint the negative sign agrees with intuition, since viscosity drives all systems towards the same state of zero motion and hence tends to increase our information about the state of the system. Nonlinear transfer contributes the positive term. This term is large for states far from equilibrium and vanishes for zero motion.

If we perform a simulation of viscous decay with an initial state sufficiently far

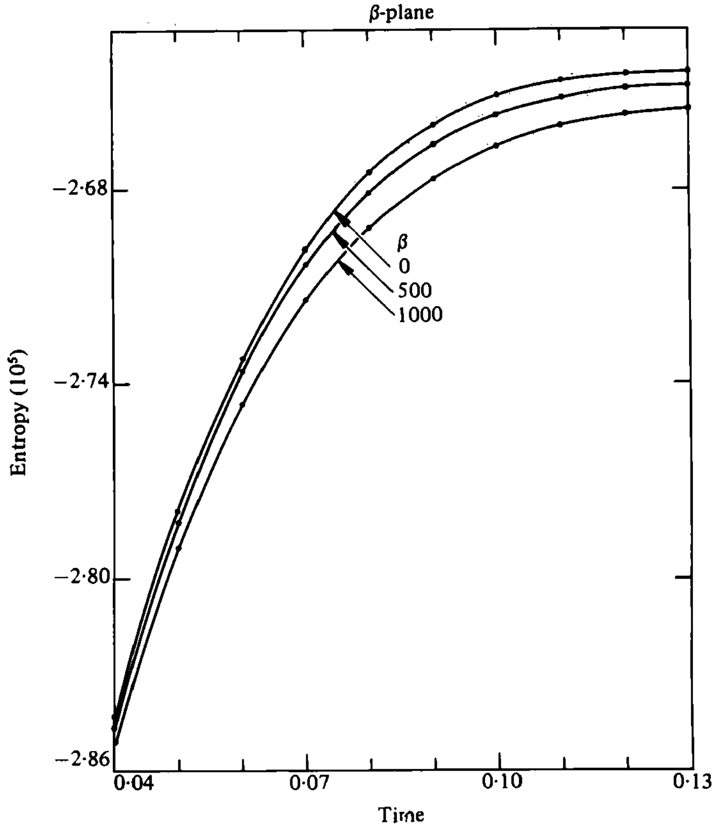

FIGURE 8. Entropy evolution on the β-plane with value of $\beta = 500$, 1000.

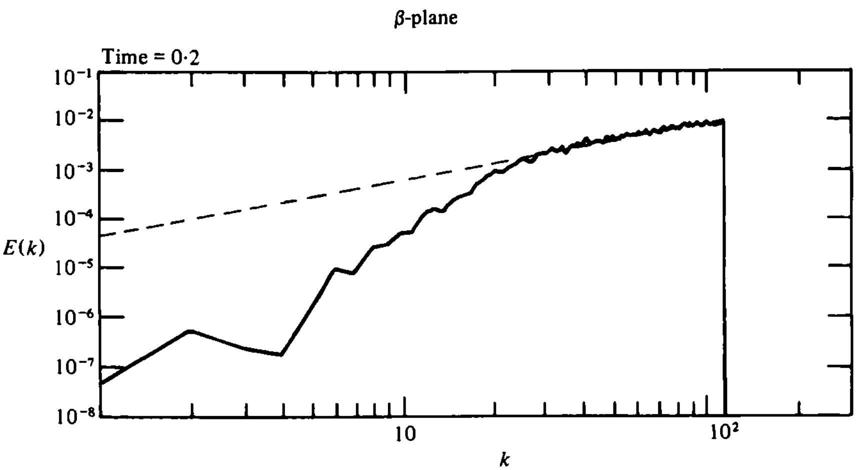

FIGURE 9. Energy spectrum at $t = 0{\cdot}2$ for β-plane simulation with $\beta = 1000$ and Rhines wavenumber $k_\beta = 25$.

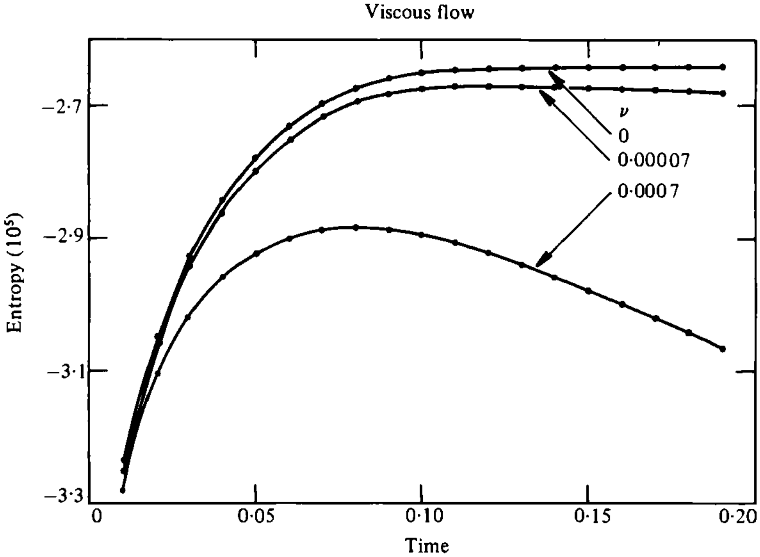

FIGURE 10. Entropy evolution in viscous simulation with values of viscosity $\nu = 0{\cdot}00007$, $0{\cdot}0007$, corresponding to initial Reynolds numbers $R = 140$, 14 respectively.

from equilibrium so that nonlinear transfer dominates, then the early evolution will be characterized by increasing S_e. A transition to decreasing S_e will eventually occur. The rate at which S_e then decreases will depend on the shape of the spectrum. Using an initial spectrum such as in figure 1 that has only high wavenumbers excited we would expect to see a transition to linear decay of S_e representing the exponential decay of the high-wavenumber energy

$$S \approx \tfrac{1}{2}\sum_{\mathbf{k}} \ln U^0_{\mathbf{k}} \exp(-2\nu k^2 t) = \tfrac{1}{2}\sum_{\mathbf{k}} \ln U^0_{\mathbf{k}} - t\nu \sum_{\mathbf{k}} k^2.$$

In such a linear decay regime the decay rate would approximate

$$dS/dt \approx -\nu \sum_{k \leqslant k_{\max}} k^2.$$

Whether such a linear decay regime will be observed depends on the shape of the spectrum and the Reynolds number of the flow. In figure 10 we show the results of two simulations using the initial conditions represented in figure 1. The values of the viscosity are $\nu = 0{\cdot}00007$ and $\nu = 0{\cdot}0007$, corresponding to viscous decay times $\tau_\nu = 1/\nu k^2$, satisfying $1 \leqslant \tau_\nu$ and $0{\cdot}1 \leqslant \tau_\nu$ respectively. The corresponding initial Taylor-based Reynolds numbers are $R \approx 140$ and 14, respectively. The figure shows a sharper earlier transition to decay for the flow with the large viscosity, $\nu = 0{\cdot}0007$, and for this viscosity a nearly linear decay with rate $-221\,000$ at $t = 0{\cdot}19$, just short of the pure viscous-decay rate $-227\,864$.

6. Discussion

We have presented a series of results that display the behaviour of the functional

$$S_e \equiv \tfrac{1}{2}\sum_{\mathbf{k}} \ln k^2 |\psi_{\mathbf{k}}|^2$$

in two-dimensional simulation. These confirm that S_e plays a role in two-dimensional flow that is directly analogous to the role played by Boltzmann's entropy for particle systems. We have emphasized that the character of the near-equilibrium behaviour of S_e can be accurately predicted by simple calculations based on canonical statistics.

The value of S_e relative to its maximum and its rate of change provides a convenient characterization of the degree to which an inviscid simulation has proceeded and the extent to which equilibrium statistical mechanics applies. Of course, inviscid spectrally truncated simulation is far from the interesting physics of forced viscid turbulence; and so the applicability of these results at present seems limited to special situations. We conjecture that the concepts employed here can be usefully extended to certain more-realistic models. This should be possible in investigating phenomena for which viscosity does not play the key role. In particular this may be the case for some of the effects found in simulation of viscid flow over irregular topography (cf. Holloway 1978; Frederiksen & Sawford 1980; Carnevale *et al.* 1981). Such simulations show development of correlations between flow and topography on timescales short compared with viscous-decay times. Perhaps this can be meaningfully interpreted in terms of entropy maximization. A further possible extension may be found of use in classical predictability studies as suggested by Carnevale & Holloway (1982).

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